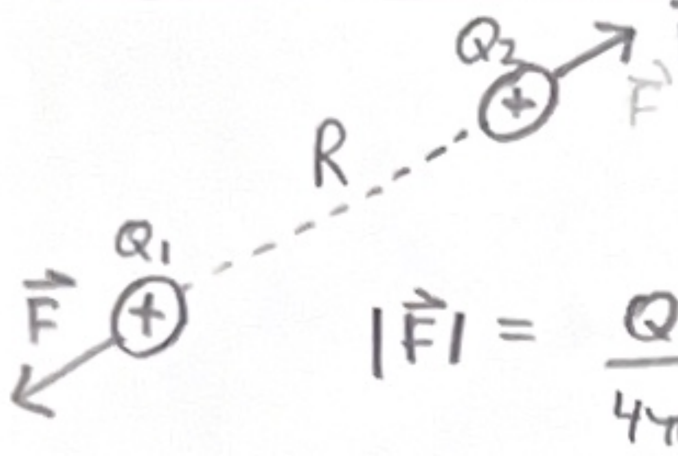
ELEC 211 Math 264

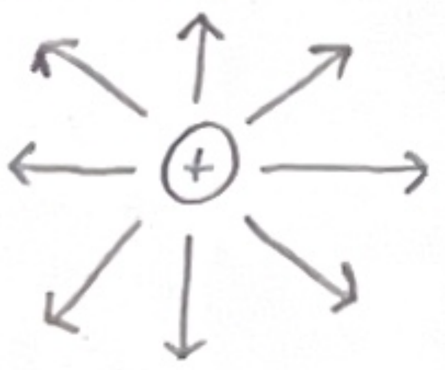
Coulomb's Law



$$|\vec{F}| = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$$

Electric Fields

Point Charge



$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}$$

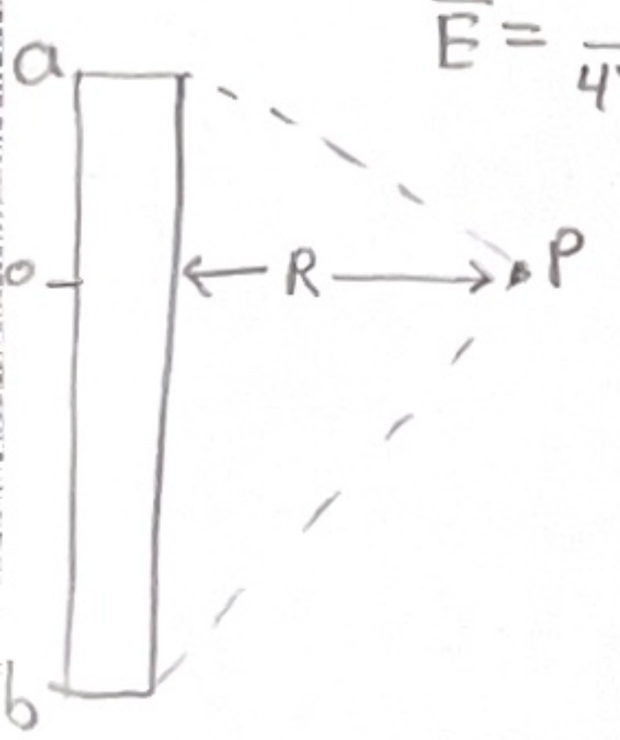
Line of Charge

Infinite:

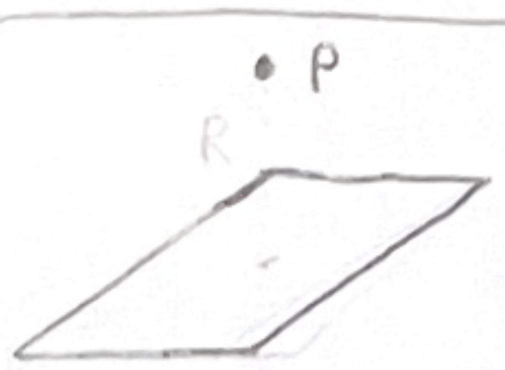
$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0 R}$$

Finite:

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0 \cdot R}\left[\frac{b}{\sqrt{R^2+b^2}} + \frac{a}{\sqrt{R^2+a^2}}\right]$$



Sheet of Charge (Infinite)



$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}$$

Work

"$\vec{E}$ field exerts a force on charged particles"

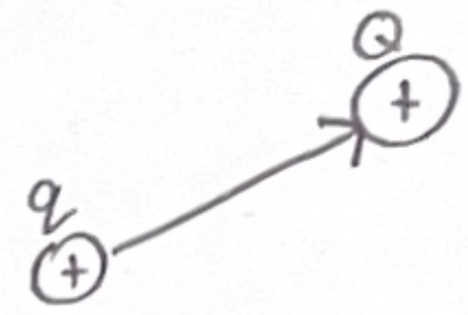
$$F = QE$$

$$W = -q\int_{initial}^{final} \vec{E}\cdot d\vec{L}$$

"To move this particle we need to exert a force, move in direction $\hat{a}_L$"

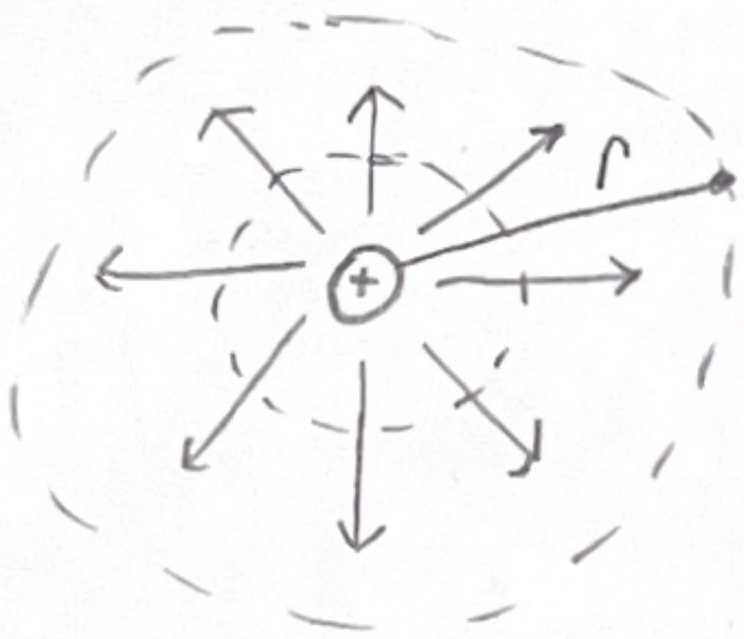
$$F = -Q\vec{E}\cdot\hat{a}_L$$

Move q towards Q



$$W = \frac{qQ}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Electric Potential



$$V = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r} \quad \text{(scalar)}$$

"Constant around equipotential sphere"

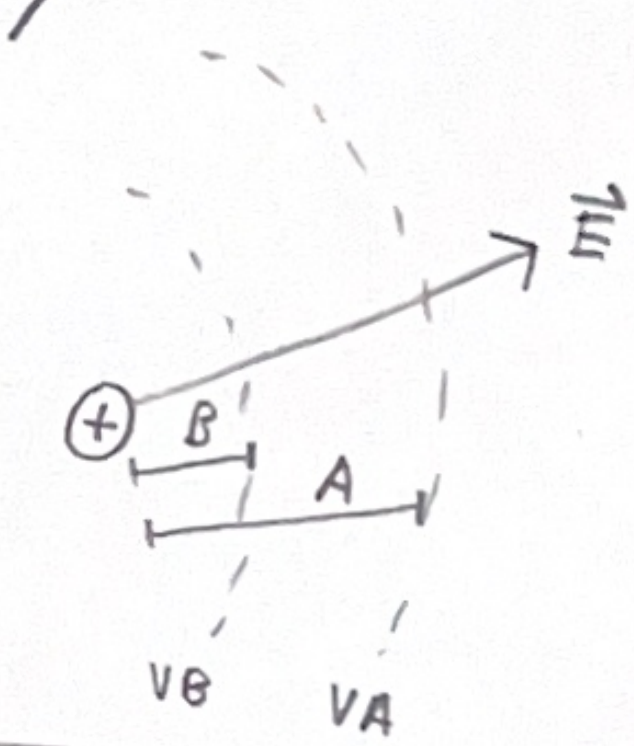
Electric field from Potential

$$\vec{E} = -\vec{\nabla}V$$

$$\vec{E} = -\left\langle \hat{i}\frac{d}{dx}(V), \hat{j}\frac{d}{dy}(V), \hat{k}\frac{d}{dz}(V)\right\rangle$$

Potential from Electric Field

$$\Delta V = V_A - V_B = -\int_B^A \vec{E}\cdot d\vec{s}$$



Finding Potential from Conservative Vector Field

*If F is conservative, there exists f where $\nabla f = \langle f_x, f_y, f_z\rangle = \vec{F}$

Ex: $\vec{F} = \langle 2x+yz, xz, yx\rangle$

$\langle f_x, f_y, f_z\rangle = \langle 2x+yz, xz, yx\rangle$

$f_x = 2x+yz \rightarrow \int (2x+yz)\,dx = x^2+xyz+g(y,z)$

$f_y = xz$

$f_z = yx$

$f = x^2+xyz+g(y,z)$

$f_y = xz + g_y(y,z)$

$\therefore g_y(y,z) = 0$

$\int 0\,dy = h(z)$

$\therefore g(y,z) = h(z)$

$f = x^2+xyz+h(z)$

$f_z = xy + h'(z)$

$\therefore h'(z) = 0$

$\int 0\,dz = C$

$\therefore h(z) = C$

$\therefore f = x^2+xyz+C$

$\nabla f = \langle 2x+yz, xz, xy\rangle = \vec{F}$ ✓

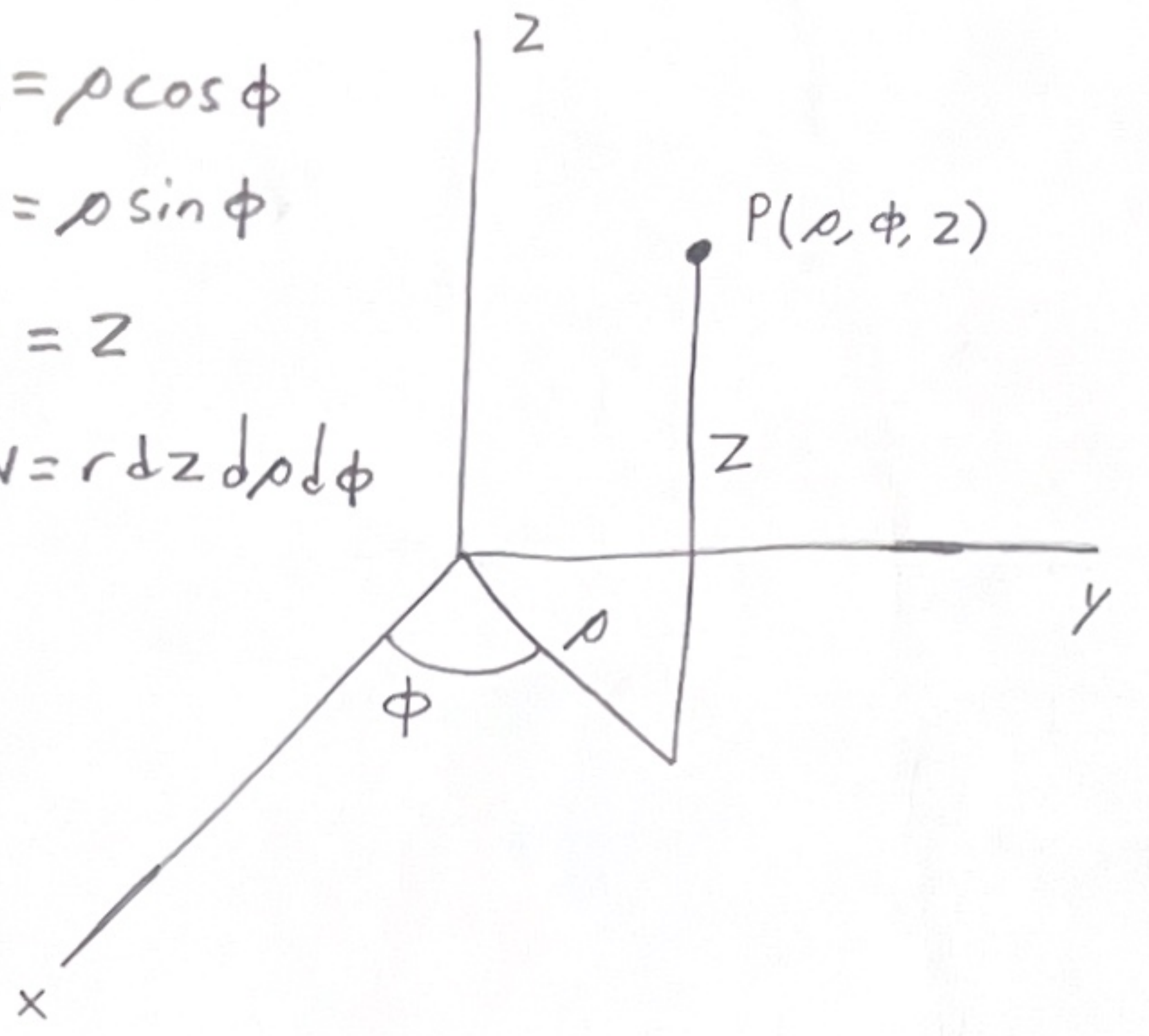
Cylindrical Coordinate System

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

$dV = r\,dz\,d\rho\,d\phi$



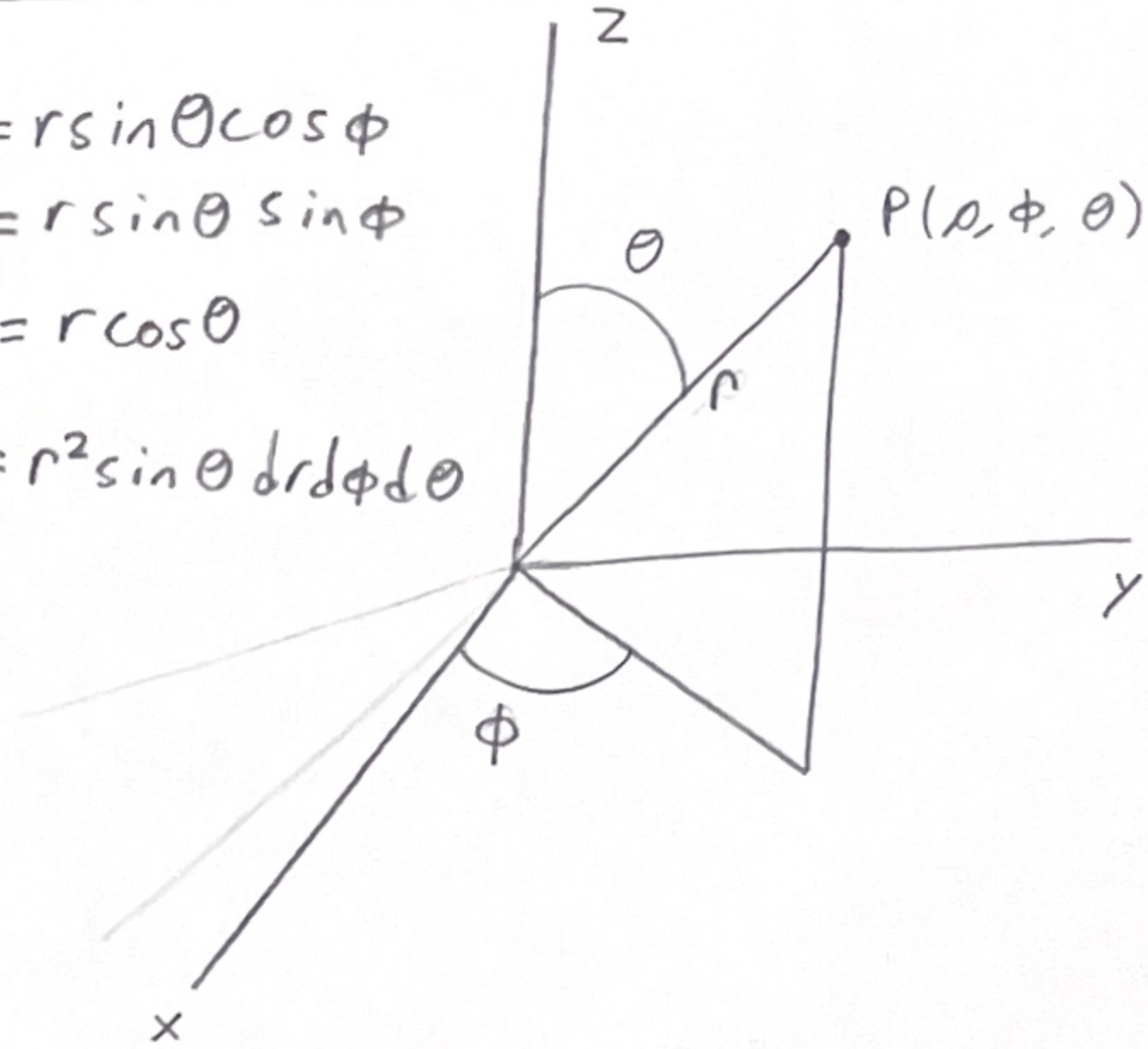
Spherical Coordinate System

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dV = r^2\sin\theta\,dr\,d\phi\,d\theta$



Electric Flux

$\Psi = Q_{encl}$

Flux Density:

$|\vec{D}| = \dfrac{\Psi}{\text{area}}$

$\vec{D} = \epsilon\vec{E}$

$\Psi = \oint\!\!\oint_S \vec{D}\cdot\vec{ds}$

Where $\vec{ds}$ is $(\hat{n})$ to surface

*if we have a surface parameterized by $r(u,v)$

then: $\vec{ds} = \dfrac{d\vec{r}}{du} \times \dfrac{d\vec{r}}{dv}$

Line Integrals

$$\int_C \vec{F}\cdot dr = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

where the curve is given by

$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k} \quad a \le t \le b$

Fundamental thereom: (Conservative)

if C, given by $\vec{r}(t)$, is smooth and $\vec{F}$ has potential function f such that $\nabla f = \vec{F}$.

$$\int_C \nabla f\cdot dr = f(\vec{r}(b)) - f(\vec{r}(a))$$

Guass's Divergence Thereom

$$\oint\!\!\oint_S \vec{D}\cdot\vec{n}\,ds = Q_{enc}$$

$$\iiint_R (\vec{\nabla}\cdot\vec{D})\,dv = \iiint_R \rho_v\,dv$$

*All equivilant

Parameterizations:

Ellipse $\rightarrow$ $x = a\cos(t)$, $y = b\sin(t)$ $\quad 0 \le t \le 2\pi$

Intersection of Functions $y = f(x)$, $z = g(x)$ $\rightarrow$ $x = t$, $y = f(t)$, $z = f(t)$ choose bounds

Line Segment

(x_0, y_0, z_0) to (x_1, y_1, z_1) $\quad 0 \le t \le 1$

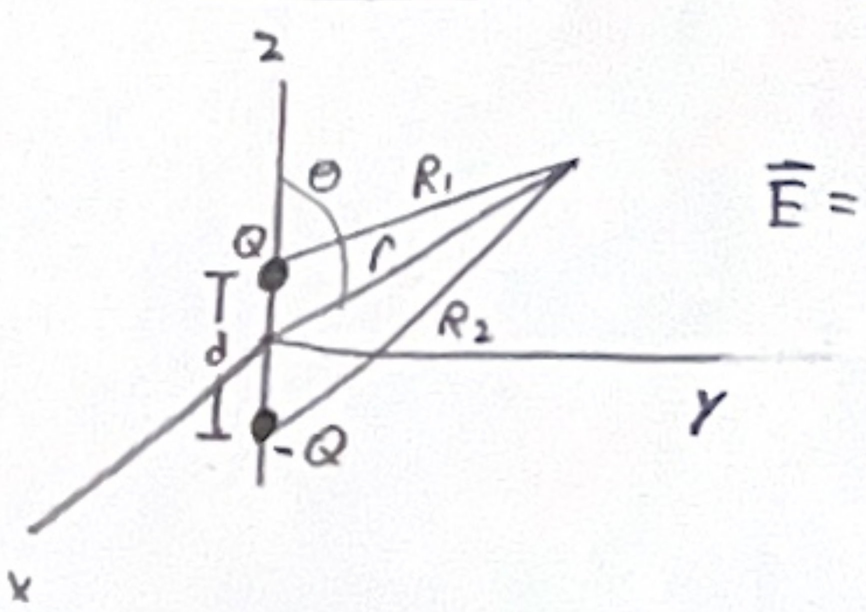
$\vec{r}(t) = (1-t)(x_0, y_0, z_0) + t(x_1, y_1, z_1)$

Surface Integrals

Surface given by $r(u,v)$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,dA$$

Dipoles



$V = \dfrac{Qd\cos\theta}{4\pi\epsilon_0 r^2}$

$\vec{E} = -\nabla V$

Dipole Moment Vector

let $p = Qd$

$d\cdot\hat{a}_r = d\cos\theta$

$V = \dfrac{p\cdot\hat{a}_r}{4\pi\epsilon_0 r^2}$

$V = \dfrac{1}{4\pi\epsilon_0 |r-r'|^2}\, p\cdot\dfrac{r-r'}{|r-r'|}$

Current

$I = \frac{dQ}{dt}$

(Current Density):

$J = (A/m^2)$

$J = \underbrace{\sigma}_{\text{conductivity}} E$

Point Form Ohm's Law

*if parallel

$V = \frac{L}{\sigma S} \cdot I = IR$

$J = \rho_v \vec{V}$

Resistance

$R = \frac{L}{\sigma S} = \rho \frac{L}{S}$

where $\rho = \frac{1}{\sigma}$

$\boxed{I = \iint_S \vec{J} \cdot d\vec{s}}$

general case:

$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \vec{E} \cdot d\vec{l}}{\iint_S \vec{J} \cdot d\vec{s}}$

Polarization

$P = \frac{\vec{P}}{V}$ where: $\vec{p} = q\vec{d}$

V = volume

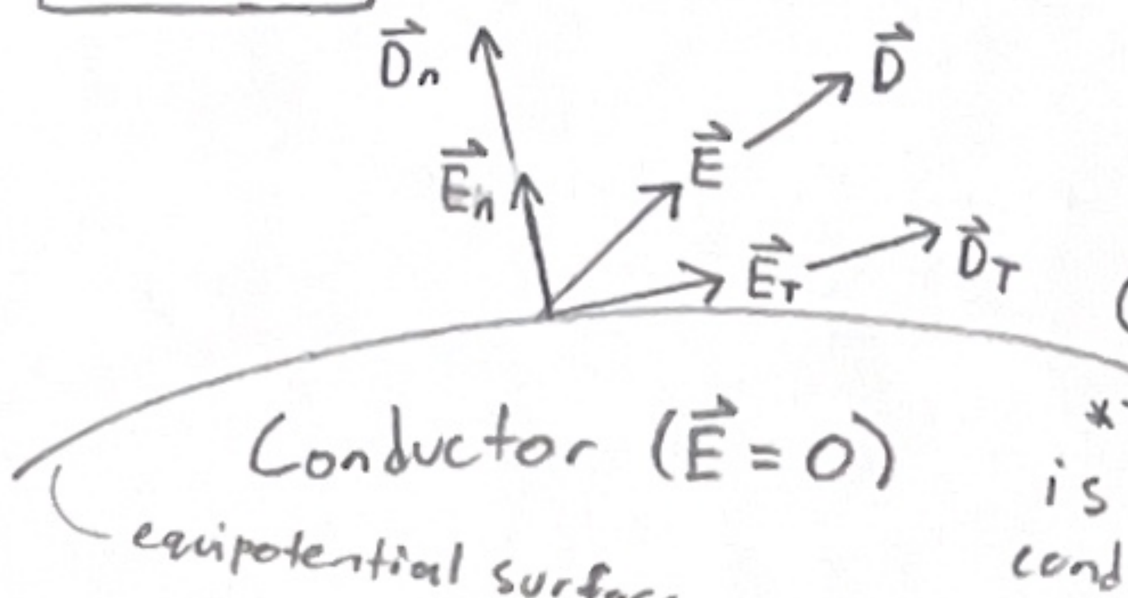
Boundary Conditions

* Use to find the field on one side of the boundary if we know the field on the other side

Cases:

1. Dielectric (ε_{r1}) and Dielectric (ε_{r2})
2. Conductor and Dielectric
3. Conductor and free space

Case 3: (Conductor and Free Space)



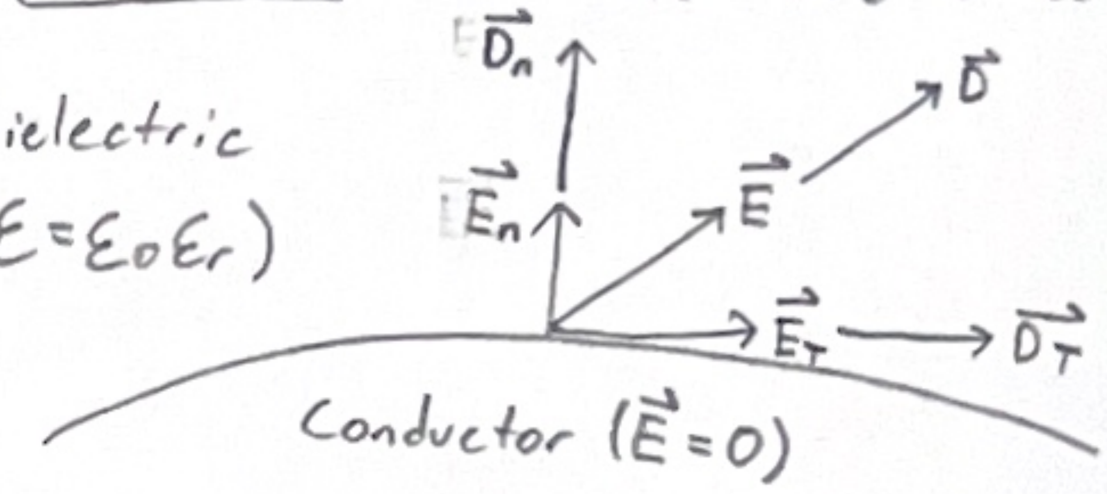
Characteristics:

1. $E_T = D_T = 0$
2. $\varepsilon_0 E_n = D_n = \rho_s$

*The electric field is external to the conductor and normal to its surface

Case 2: (Conductor and Dielectric)

Dielectric ($\varepsilon = \varepsilon_0 \varepsilon_r$)

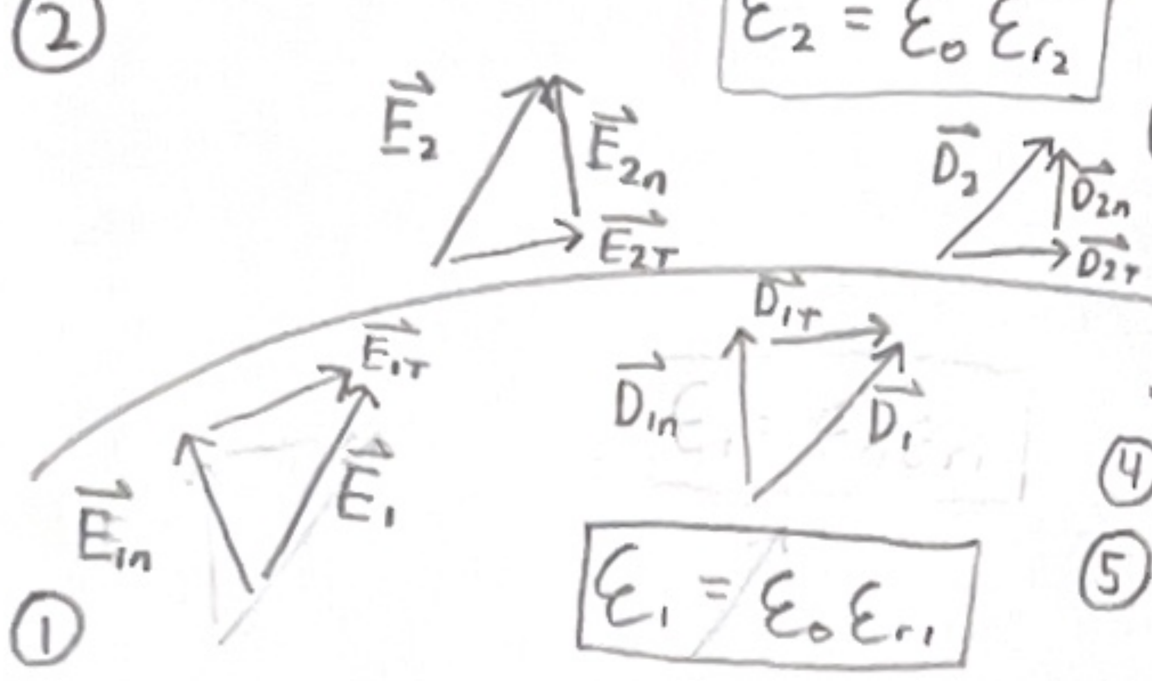


Characteristics:

1. $E_T = D_T = 0$
2. $\varepsilon_0 \varepsilon_r E = D_n = \rho_s$

Case 1: (Dielectric and Dielectric)

$\boxed{\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}}$



$\boxed{\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}}$

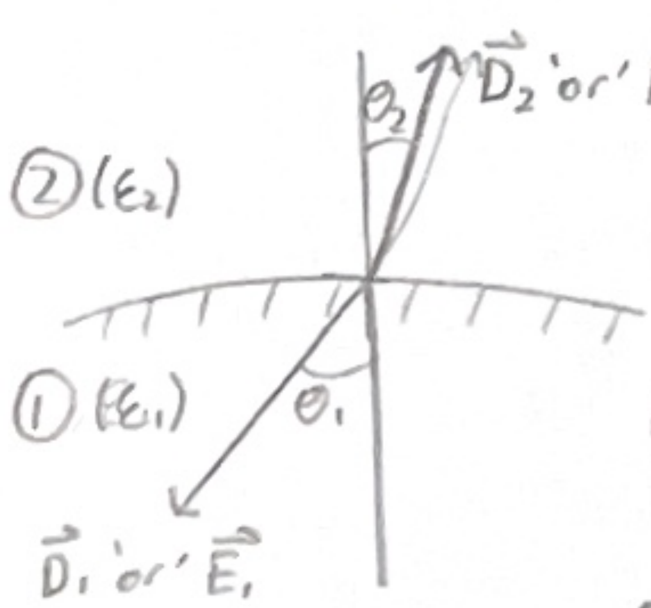
Characteristics:

1. $E_{1T} = E_{2T}$
2. $\frac{D_{1T}}{\varepsilon_1} = \frac{D_{2T}}{\varepsilon_2}$
3. $D_{2n} - D_{1n} = \rho_s$

*if $\rho_s = 0$ (@ Boundary)

4. $D_{2n} = D_{1n}$
5. $\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$

Refraction



Characteristics:

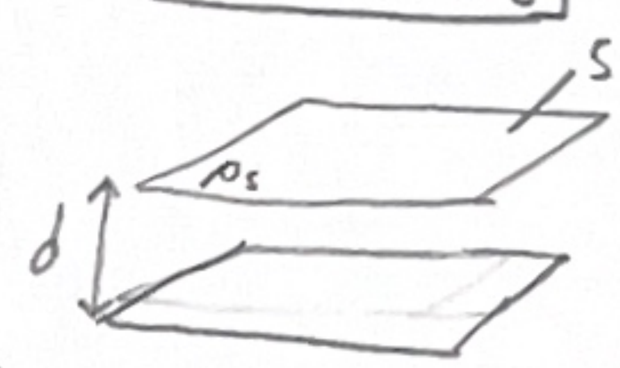
$\left\{ \begin{array}{l} E_{1T} = E_{2T} \\ D_{1n} = D_{2n} \end{array} \right\}$

1. $E_1 \sin\theta_1 = E_2 \sin\theta_2$
2. $D_1 \cos\theta_1 = D_2 \cos\theta_2$

$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$

3. $\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$

Capacitors

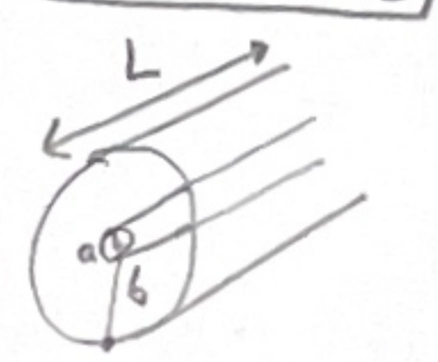


$C = \frac{Q}{V}$

$C = \frac{\varepsilon_0 S}{d}$

(Parallel-Plate, constant ρ_s)

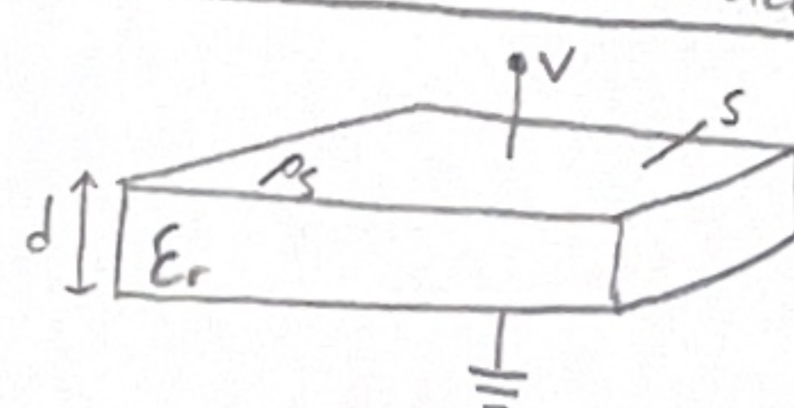
Coaxial Cable



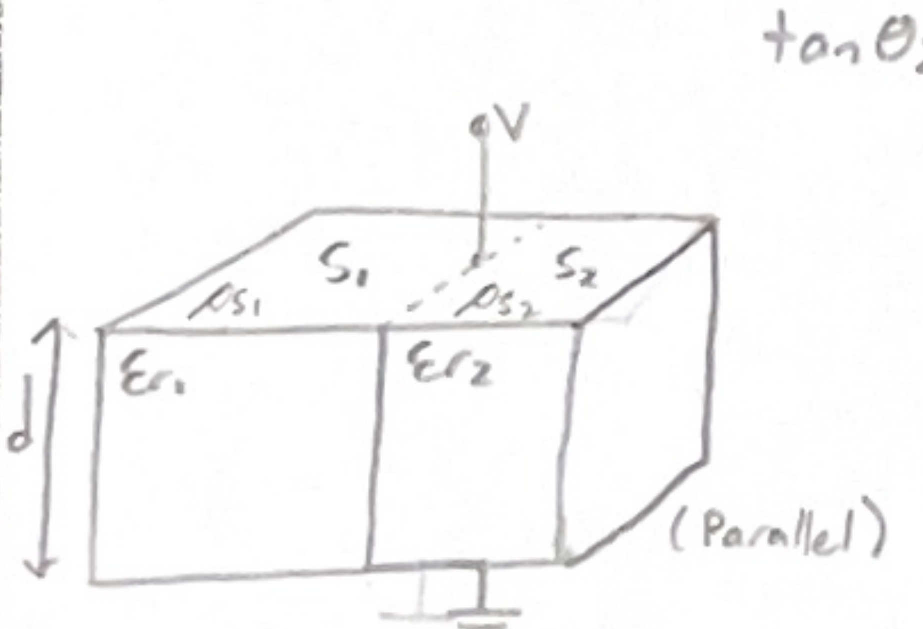
$C = \frac{Q}{V}$

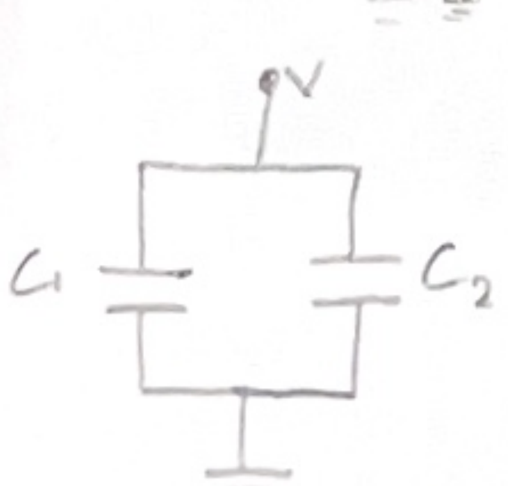
$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$

Capacitor with Dielectrics



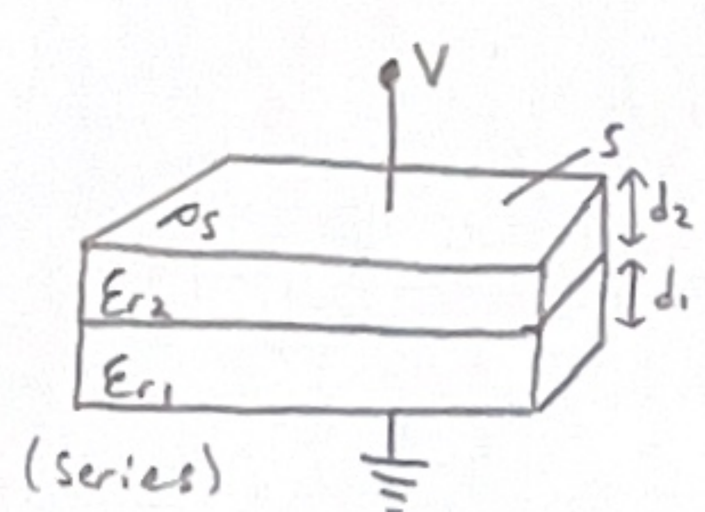
$C = \frac{\varepsilon_0 \varepsilon_r S}{d}$





$C = \frac{\varepsilon_{r1} \varepsilon_0 S_1}{d} + \frac{\varepsilon_{r2} \varepsilon_0 S_2}{d}$

$= \frac{\varepsilon_0}{d} \left(\varepsilon_{r1} S_1 + \varepsilon_{r2} S_2 \right)$



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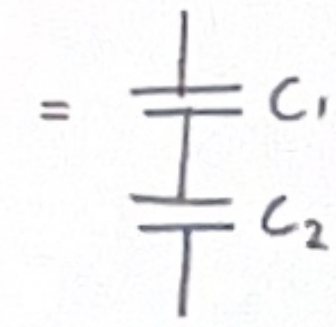


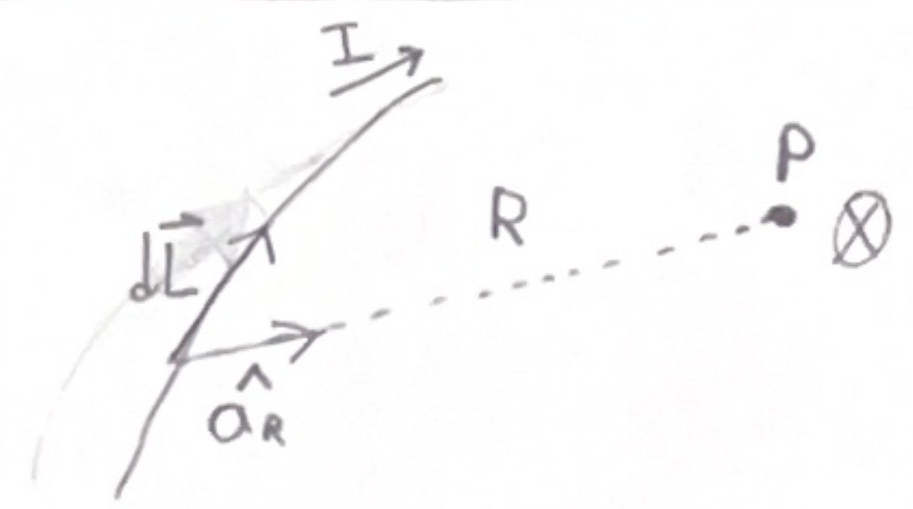
$C_1 = \frac{\varepsilon_{r1} \varepsilon_0 S}{d_1}$

$C_2 = \frac{\varepsilon_{r2} \varepsilon_0 S}{d_2}$

$C = \frac{\varepsilon_{r1} \varepsilon_{r2} \varepsilon_0 S}{\varepsilon_{r2} d_1 + \varepsilon_{r1} d_2}$

Graded Dielectric

Biot-Savart Law



$$d\vec{H} = \frac{I d\vec{L} \times \hat{a}_R}{4\pi R^2}$$

$$\vec{H} = \oint \frac{I d\vec{L} \times \hat{a}_R}{4\pi R^2}$$

Surface Current

$$\vec{H} = \iint_S \frac{\vec{K} \times \hat{a}_R \, dS}{4\pi R^2}$$

Volume Current

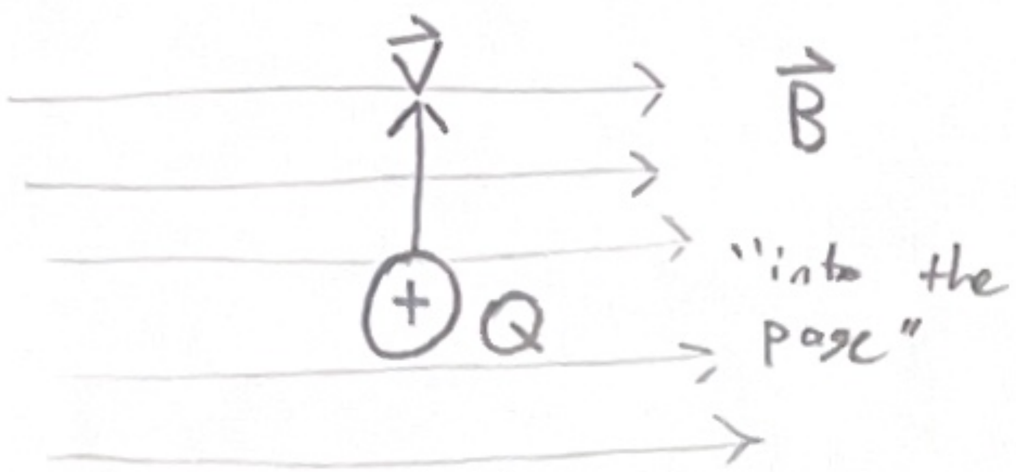
$$\vec{H} = \iiint_{vol} \frac{J \times \hat{a}_R \, dV}{4\pi R^2}$$

Divergence Thereom

$$\Phi = \oiint_S \vec{B} \cdot d\vec{S} = 0$$

$$\therefore \nabla \cdot \vec{B} = 0$$

Force on a Moving Charge



$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

Magnetic Flux Density

$\vec{B} = \mu_0 \vec{H}$ where:

$\mu_0 = 4\pi \times 10^{-7}$ H/m

$$\Phi = \iint_S \vec{B} \cdot d\vec{S}$$

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PHYSICAL CONSTANTS

Permittivity of free space:	$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m	Permeability of free space:	$\mu_0 = 4\pi \times 10^{-7}$ H/m
Electron charge:	$e = 1.602 \times 10^{-19}$ C	Electron mass:	$m = 9.109 \times 10^{-31}$ kg
Speed of light in vacuum:	$c = 2.998 \times 10^{8}$ m/s		

ELECTROSTATIC PRINCIPLES

Coulomb's Law: $\mathbf{F}_2 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{R_{12}^2}\mathbf{a}_{12}$ $\qquad \mathbf{a}_{12} = \dfrac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \dfrac{\mathbf{R}_{12}}{R_{12}} = \dfrac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$

Point Charge Q at O: $\mathbf{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\mathbf{a}_r, \; V = \dfrac{Q}{4\pi\varepsilon_0 r}$ $\qquad$ (r comes from spherical coords)

Line Charge, density ρ_L, on z-axis: $\mathbf{E} = \dfrac{\rho_L}{2\pi\varepsilon_0}\left(\dfrac{\mathbf{a}_\rho}{\rho}\right), \; V = \dfrac{\rho_L}{2\pi\varepsilon_0}\ln\left(\dfrac{1}{\rho}\right)$ $\qquad$ (ρ comes from cylindrical coords)

Sheet Charge, density ρ_S, on $z = 0$: $\mathbf{E} = \pm\dfrac{\rho_S}{2\varepsilon_0}\mathbf{a}_z, \; V = -\dfrac{\rho_S |z|}{2\varepsilon_0}$ $\qquad$ (Both ρ_S and ρ_L must be constant here.)

Electric Flux Density: (C/m^2) $\mathbf{D} = \varepsilon\mathbf{E}$ $\qquad$ ($\varepsilon = \varepsilon_0\varepsilon_r$ in general; $\varepsilon_r = 1$ in free space)

Gauss's Law, I: (C) $Q_{enc} = \Psi$, where $\qquad \Psi = \oiint_S \mathbf{D} \bullet \hat{\mathbf{n}}\, dS$ is net outward flux

Gauss's Law, II: $Q_{enc} = \iiint_{\mathcal{V}} \rho_v \, dv$, where $\qquad \rho_v = \nabla \bullet \mathbf{D}$ gives charge density

Electric field and potential: (V/m) $\mathbf{E} = -\nabla V$ $\qquad V(B) - V(A) = -\displaystyle\int_A^B \mathbf{E} \bullet d\mathbf{L}$ (path indep)

Generalized Poisson Equation: $\nabla \bullet (\varepsilon \nabla V) = -\rho_v$ $\qquad$ (Case $\rho_v = 0$, $\varepsilon =$ const is Laplace's Equation.)

Energy in Electrostatic Field: $W_E = \frac{1}{2}\iiint_{\mathcal{R}} \mathbf{D} \bullet \mathbf{E}\, dv = \frac{1}{2}\iiint_{\mathcal{R}} \varepsilon |\mathbf{E}|^2 \, dv$

CONDUCTORS, CURRENT, RESISTANCE

Ideal conductor ("$\sigma \to \infty$"): $\mathbf{E}_T = 0$ $\qquad V =$ const.

Ideal conductor boundary: $\mathbf{E} \parallel \hat{\mathbf{n}}$ $\qquad \rho_S = \mathbf{D} \bullet \hat{\mathbf{n}}$

Current and conductivity: $\mathbf{J} = \sigma\mathbf{E}$ "Ohm's Law I" $\qquad I = \iint_S \mathbf{J} \bullet \hat{\mathbf{n}}\, dS$

$\mathbf{J} = \rho_v \mathbf{v}$ $\qquad \nabla \bullet \mathbf{J} = -\dfrac{\partial \rho_v}{\partial t}$

Simple Resistor (length L, constant cross-section S, constant conductivity σ): $R = \dfrac{L}{\sigma S}$

Fancy Resistor (all current from A to B crosses surface $\mathcal{S}$ "Ohm's Law II"): $R = \dfrac{|\Delta V|}{|I|} = \dfrac{\left| -\displaystyle\int_A^B \mathbf{E} \bullet d\mathbf{L} \right|}{\left| \displaystyle\iint_S \mathbf{J} \bullet \hat{\mathbf{n}}\, dS \right|}$

CAPACITORS AND DIELECTRICS

Permittivity: $\varepsilon = \varepsilon_r \varepsilon_0$ $\qquad$ Coaxial: $\qquad$ (Gauss's Law still works, as above)

Polarization: $\mathbf{P} = \mathbf{D} - \varepsilon_0 \mathbf{E}$ $\qquad \dfrac{C}{L} = \dfrac{2\pi\varepsilon}{\ln(b/a)}$

Simple Capacitor (parallel plates of area S, separation d): $C = \dfrac{\varepsilon S}{d}$ stores $W_E = \frac{1}{2}CV^2$ Joules

Fancy Capacitor (surface $\mathcal{S}$ is one plate; points A, B on opposite plates): $C = \dfrac{|Q|}{|\Delta V|} = \dfrac{\left| \displaystyle\iint_S \mathbf{D} \bullet \hat{\mathbf{n}}\, dS \right|}{\left| -\displaystyle\int_A^B \mathbf{E} \bullet d\mathbf{L} \right|}$

Dielectric interface with normal $\mathbf{n}$: $\mathbf{D}_1 \bullet \mathbf{n} = \mathbf{D}_2 \bullet \mathbf{n}$ AND $\mathbf{E}_1 \times \mathbf{n} = \mathbf{E}_2 \times \mathbf{n}$

$D_{1N} = D_{2N}$ $\qquad E_{1T} = E_{2T}$

$\dfrac{\tan(\theta_1)}{\tan(\theta_2)} = \dfrac{\varepsilon_1}{\varepsilon_2}$

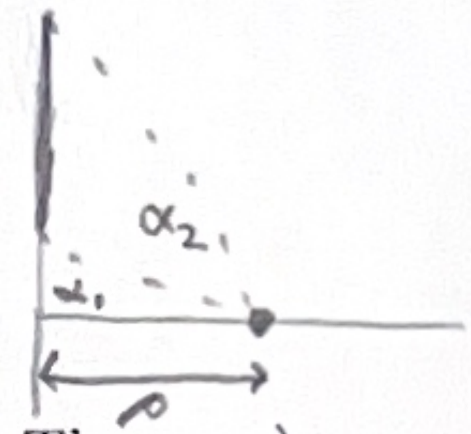
FPL: $\Phi = \int_{z=0}^{L} \int_{\rho=a}^{b} B \, d\rho \, d\phi$

MAGNETOSTATICS

Biot-Savart Law: $d\mathbf{H} = \dfrac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$ $\qquad \mathbf{H} = \displaystyle\int \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$

Current I flowing in filament $\rho = 0$, direction $\mathbf{a}_z$: $\mathbf{H} = \dfrac{I}{2\pi\rho}\mathbf{a}_\phi$; or, for segment, $\mathbf{H} = \dfrac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\,\mathbf{a}_\phi$

Current sheet with density $\mathbf{K}$ [A/m], normal $\hat{\mathbf{n}}$: (A/m) $\mathbf{H} = \frac{1}{2}\mathbf{K} \times \hat{\mathbf{n}}$ $\qquad I = \displaystyle\int \mathbf{K} \bullet d\mathbf{w}$



Current crossing surface $\mathcal{S}$, from current density $\mathbf{J}$: $I = \displaystyle\iint_{\mathcal{S}} \mathbf{J} \bullet d\mathbf{S}$ $\qquad \mathbf{J} = \nabla \times \mathbf{H}$

Ampère's Circuital Law (ACL): $I = \displaystyle\oint \mathbf{H} \bullet d\mathbf{L}$ $\qquad$ (compare Stokes's Theorem)

Magnetic Flux Density: $\left[\mathbf{B} = \frac{\Phi}{S}\right]$ (Wb/m²) $\mathbf{B} = \mu\mathbf{H}$ $\qquad \mu = \mu_r\mu_0$

Magnetic Flux (Wb): (Wb) $\Phi = \displaystyle\iint_{\mathcal{S}} \mathbf{B} \bullet d\mathbf{S}$ $\qquad \displaystyle\oiint_{\mathcal{S}} \mathbf{B} \bullet d\mathbf{S} = 0$

Energy in Steady Magnetic Field: $W_H = \frac{1}{2}\displaystyle\iiint_{\mathcal{R}} \mathbf{B} \bullet \mathbf{H}\,dv = \frac{1}{2}\iiint_{\mathcal{R}} \mu\,|\mathbf{H}|^2\,dv$

Magnetic Force on Moving Charge: (N) $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$ $\qquad \mathbf{J} = \mathbf{v}\rho_v$

Magnetic Force on Current Filament: $d\mathbf{F} = I\,d\mathbf{L} \times \mathbf{B}$ $\qquad \mathbf{F} = \displaystyle\int_C I\,d\mathbf{L} \times \mathbf{B} = -\int_C I\mathbf{B} \times d\mathbf{L}$

Magnetic Force on Current Sheet or Cloud: $d\mathbf{F} = (\mathbf{K}\,dS) \times \mathbf{B}$ $\qquad d\mathbf{F} = (\mathbf{J}\,dv) \times \mathbf{B}$

Magnetic Dipole Moment ($\mathbf{m} = \mathbf{p}_m$): $d\mathbf{m} = I\,d\mathbf{S}$ $\qquad \mathbf{m} = NIS\hat{\mathbf{n}}$

Magnetic Torque on Given Dipole: $\vec{\tau} = \mathbf{m} \times \mathbf{B}$ $\qquad |\vec{\tau}| = NI\,|\mathbf{B}|\,|\mathbf{S}|$, if $\mathbf{B} \perp \mathbf{S}$

Review: Force $\mathbf{F}$ with moment arm $\mathbf{R}$ gives torque: $\vec{\tau} = \mathbf{R} \times \mathbf{F}$

INDUCTORS AND MAGNETIC MATERIALS

Permeability: $\mu = \mu_r\mu_0$

Simple inductor (N filaments, current I in each): $L = \dfrac{N\Phi}{I}$ $\qquad$ stores $W_H = \frac{1}{2}LI^2$ Joules

Mutual Inductance: $M_{12} = \dfrac{N_2\Phi_{12}}{I_1} = \dfrac{N_1\Phi_{21}}{I_2} = M_{21}$

Material interface with normal $\mathbf{n}$: $\mathbf{B}_1 \bullet \mathbf{n} = \mathbf{B}_2 \bullet \mathbf{n}$ $\qquad \mathbf{H}_1 \times \mathbf{n} = \mathbf{H}_2 \times \mathbf{n}$

MAGNETIC CIRCUITS

Magnetomotive force (simple setup N turns, current I): $V_m = NI$

Magnetomotive force (general filament from A to B): $V_m(B) - V_m(A) = -\displaystyle\int_A^B \mathbf{H} \bullet d\mathbf{L}$ $\qquad$ (path restrictions apply)

Reluctance (cross-section S, length ℓ): $\mathcal{R} = \dfrac{V_m}{\Phi} = \dfrac{\ell}{\mu S}$ $\qquad$ (integral defining Φ shown above)

Air-gap force (cross-section S): $\mathbf{F} = \dfrac{1}{2\mu_0}|\mathbf{B}|^2 S\,\hat{\mathbf{n}}$

MAXWELL'S EQUATIONS (POINT FORM, GENERAL CASE set $\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$ and $\frac{\partial \mathbf{D}}{\partial t} = \mathbf{0}$ in static situations)

$\boxed{\nabla \bullet \mathbf{D} = \rho_v}$ $\qquad \nabla \bullet \mathbf{B} = 0$ $\qquad \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ $\qquad \boxed{\nabla \times \mathbf{H} = \mathbf{J}} + \dfrac{\partial \mathbf{D}}{\partial t}$

TIME-VARYING FIELDS

Faraday's Law (case of $N = 1$ current filament): $\text{emf} = -\dfrac{d\Phi}{dt} = -\displaystyle\iint_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \bullet \hat{\mathbf{n}}\,dS$ $\qquad$ (units: Volts)

$\text{emf} = \displaystyle\oint_C \mathbf{E} \bullet d\mathbf{L}$ $\qquad$ (loop shape matters!)

$\text{emf} = BLv\sin\theta$

VECTOR IDENTITIES

For $\mathbf{u} = u_x\mathbf{a}_x + u_y\mathbf{a}_y + u_z\mathbf{a}_z$, $\mathbf{v} = v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z$, $\mathbf{w} = w_x\mathbf{a}_x + w_y\mathbf{a}_y + w_z\mathbf{a}_z$,

$$\mathbf{u} \bullet \mathbf{v} = u_xv_x + u_yv_y + u_zv_z = |\mathbf{u}|\,|\mathbf{v}|\cos(\theta), \quad 0 \le \theta \le \pi$$

$$|\mathbf{u}| = \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \langle u_yv_z - u_zv_y,\ u_zv_x - u_xv_z,\ u_xv_y - u_yv_x \rangle$$

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$$

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}$$

DISTANCES AND PROJECTIONS

From point (x_0, y_0, z_0) to plane $Ax + By + Cz = D$: $\quad s = \dfrac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$

$$\mathbf{F} = \mathbf{proj_u}(\mathbf{F}) + \mathbf{orth_u}(\mathbf{F}) \qquad \mathbf{proj_u}(\mathbf{F}) = \left(\frac{\mathbf{F} \bullet \mathbf{u}}{\mathbf{u} \bullet \mathbf{u}}\right)\mathbf{u}$$

DERIVATIVE IDENTITIES valid for smooth scalar-valued ϕ, ψ and smooth vector-valued $\mathbf{F}$, $\mathbf{G}$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla \bullet (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \bullet \mathbf{G} - \mathbf{F} \bullet (\nabla \times \mathbf{G})$$

$$\nabla \bullet (\phi\mathbf{F}) = (\nabla\phi) \bullet \mathbf{F} + \phi(\nabla \bullet \mathbf{F})$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \bullet \mathbf{G}) - \mathbf{G}(\nabla \bullet \mathbf{F}) - (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$$

$$\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$$

$$\nabla(\mathbf{F} \bullet \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$$

$$\nabla \times (\nabla\phi) = 0 \qquad (\mathbf{curl\,grad} = 0)$$

$$\nabla \bullet (\nabla \times \mathbf{F}) = 0 \qquad (\mathrm{div}\,\mathbf{curl} = 0)$$

$$\nabla^2\phi(x,y,z) = \nabla \bullet \nabla\phi(x,y,z) = \mathrm{div}\,\mathbf{grad}\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

SURFACE NORMALS AND AREA ELEMENTS

For any oriented surface normal $\mathbf{n} \neq \mathbf{0}$, $\quad d\mathbf{S} = \widehat{\mathbf{n}}\,dS = \dfrac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{a}_z|}\,dx\,dy = \dfrac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{a}_y|}\,dx\,dz = \dfrac{\mathbf{n}}{|\mathbf{n} \bullet \mathbf{a}_x|}\,dy\,dz, \qquad dS = |d\mathbf{S}|$

Graph Surface $z = f(x,y)$: $\quad$ normal $\mathbf{n} = \pm\left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, -1 \right\rangle \qquad \widehat{\mathbf{n}}\,dS = \pm\left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, -1 \right\rangle dx\,dy$

Level Surface $G(x,y,z) = 0$: $\quad$ normal $\mathbf{n} = \pm\nabla G(x,y,z) \qquad$ (choose sign to orient)

Parametric Surface $\langle x,y,z \rangle = \mathbf{R}(u,v)$: $\quad d\mathbf{S} = \pm\left(\dfrac{\partial\mathbf{R}}{\partial u} \times \dfrac{\partial\mathbf{R}}{\partial v}\right) du\,dv \qquad$ (choose sign to orient; $\widehat{\mathbf{n}} = \dfrac{d\mathbf{S}}{|d\mathbf{S}|}$)

CARTESIAN COORDINATES (x, y, z)

Line Element: $d\mathbf{L} = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz$

Volume Element: $dv = dx\,dy\,dz$

Scalar field: $f(x,y,z)$

Vector field: $\mathbf{F}(x,y,z) = F_x\,\mathbf{a}_x + F_y\,\mathbf{a}_y + F_z\,\mathbf{a}_z$

Differential operator ∇: $\quad \nabla = \mathbf{a}_x\dfrac{\partial}{\partial x} + \mathbf{a}_y\dfrac{\partial}{\partial y} + \mathbf{a}_z\dfrac{\partial}{\partial z} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$

Gradient: $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{a}_x + \dfrac{\partial f}{\partial y}\mathbf{a}_y + \dfrac{\partial f}{\partial z}\mathbf{a}_z$

Divergence: $\nabla \bullet \mathbf{F}(x,y,z) = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$

Curl: $\nabla \times \mathbf{F} = \mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

Laplacian: $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial y^2}$

POLAR AND CYLINDRICAL COORDINATES (ρ, ϕ, z)

Transformation: $x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$

Local basis: $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y, \quad \mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y, \quad \mathbf{a}_z = \mathbf{a}_z$

Surface element (on $\rho = a$): $d\mathbf{S} = \pm a\,\mathbf{a}_\rho\,d\phi\,dz$

Line Element: $d\mathbf{L} = \mathbf{a}_\rho\,d\rho + \rho\mathbf{a}_\phi\,d\phi + \mathbf{a}_z\,dz$

Scalar field: $f(\rho, \phi, z)$

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{a}_\phi + \frac{\partial f}{\partial z}\mathbf{a}_z$$

$\rho = \sqrt{x^2+y^2}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$$\nabla \times \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

Surface element (on $z =$const.): $d\mathbf{S} = \pm\rho\,\mathbf{a}_z\,d\rho\,d\phi$

Volume element: $dv = \rho\,d\rho\,d\phi\,dz$

Vector field: $\mathbf{F}(\rho, \phi, z) = F_\rho\,\mathbf{a}_\rho + F_\phi\,\mathbf{a}_\phi + F_z\,\mathbf{a}_z$

$$\nabla \bullet \mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Solenoid:

$B = \frac{N_0 N I}{L}$

Toroid:

$B = \frac{N_0 N I}{2\pi R}$

SPHERICAL COORDINATES (r, θ, ϕ)

Transformation: $x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

Local basis: $\mathbf{a}_r = \sin\theta\cos\phi\,\mathbf{a}_x + \sin\theta\sin\phi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$, $\quad \mathbf{a}_\theta = \cos\theta\cos\phi\,\mathbf{a}_x + \cos\theta\sin\phi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$,

$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$

Volume element: $dv = r^2\sin\theta\,dr\,d\theta\,d\phi$

Line Element: $d\mathbf{L} = \mathbf{a}_r\,dr + r\mathbf{a}_\theta\,d\theta + r\sin\theta\mathbf{a}_\phi\,d\phi$

Scalar field: $f(r, \theta, \phi)$

$r = \sqrt{x^2+y^2+z^2}$

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{a}_\phi$$

$$\nabla \times \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\theta & r\sin\theta\mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

Surface area element (on $r = a$): $d\mathbf{S} = \pm a^2\sin\theta\,\mathbf{a}_r\,d\theta\,d\phi$

Vector field: $\mathbf{F}(r, \theta, \phi) = F_r\,\mathbf{a}_r + F_\theta\,\mathbf{a}_\theta + F_\phi\,\mathbf{a}_\phi$

$$\nabla \bullet \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

INTEGRATING DERIVATIVES: THE FUNDAMENTAL THEOREM OF CALCULUS (FTC)

Line-integral form: $$\int_C \nabla g \bullet d\mathbf{L} = \int_C \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy + \frac{\partial g}{\partial z}\,dz = g_{\text{final}} - g_{\text{initial}}$$

Stokes's Theorem: $$\iint_S (\nabla \times \mathbf{G}) \bullet d\mathbf{S} = \oint_C \mathbf{G} \bullet d\mathbf{L} = \oint_C G_x\,dx + G_y\,dy + G_z\,dz$$

Divergence Theorem: $$\iiint_{\mathcal{R}} \nabla \bullet \mathbf{G}\,dv = \oiint_S \mathbf{G} \bullet \hat{\mathbf{n}}\,dS$$

DEFINITE INTEGRALS

$$\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \cos x\,dx = 1 \qquad \int_0^{\pi/2} \sin^3 x\,dx = \int_0^{\pi/2} \cos^3 x\,dx = \frac{2}{3} \qquad \int_0^{\pi/2} \sin^5 x\,dx = \int_0^{\pi/2} \cos^5 x\,dx = \frac{8}{15}$$

$$\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4} \qquad \int_0^{\pi/2} \sin^4 x\,dx = \int_0^{\pi/2} \cos^4 x\,dx = \frac{3\pi}{16} \qquad \int_0^{\pi/2} \sin^6 x\,dx = \int_0^{\pi/2} \cos^6 x\,dx = \frac{5\pi}{32}$$

INDEFINITE INTEGRALS

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) \qquad \int \tan x\,dx = \ln|\sec x| \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad (a>0)$$

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x \qquad \int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \quad (a>0)$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) \quad (a>0) \qquad \int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}} \qquad \int \frac{dx}{(x^2\pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2\pm a^2}}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \quad (a>0) \qquad \int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right|$$

Adapted from R. A. Adams, *Calculus, A Complete Course*, Addison-Wesley, 2003.